

Advanced Algebra.

MA180-4.

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Outline

- 1 The Language of Mathematics: Logic and Sets.
 - Propositional Logic.
 - Valid Arguments.
 - Sets and Boolean Algebra.
 - Functions and Relations.
- 2 Examples of Algebraic Objects: Permutations and Polynomials.
 - Composition of Functions.
 - Permutations.
 - Polynomials.
 - Factorisation of Polynomials.
- 3 Mathematical Tools: Induction and Matrix Algebra.
 - Mathematical Induction.
 - Examples and Applications of Induction.
 - Determinants.
 - Eigenvalues and Eigenvectors.

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.





Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

References.

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-  Lindsay Childs.
A Concrete Introduction to Higher Algebra.
Springer 2000.
-  Douglas E. Ensley and J.Winston Crawley
Discrete Mathematics.
Wiley 2006.
-  Mark V. Lawson
Algebra & Geometry: An Introduction to University Mathematics
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Introduction: The Language of Mathematics Mathematics ...

- ... is about solving **problems**.
- ... explains **patterns**.
- ... is a set of statements deduced **logically** from axioms and definitions.
- ... uses **abstraction** to model the real world.
- ... employs a precise and powerful **language** to organize, communicate, and manipulate ideas.

As with any language, in order to participate in a conversation, it helps to be able to **read** and **write**. In this section, we introduce basic elements of the mathematical language and study their meaning:

- **logic**: the language of mathematical arguments;
- **sets**: the language of relationships between mathematical objects.

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

Links: The Language of Mathematics.

- http://en.wikipedia.org/wiki/Mathematics_as_a_language
- http://en.wikipedia.org/wiki/Knights_and_Knaves
- <http://www.raymondsmullyan.com>
- <http://www.iep.utm.edu/prop-log/>
- http://en.wikipedia.org/wiki/Mathematical_proof
- <http://plato.stanford.edu/entries/boolalg-math/>
- http://en.wikipedia.org/wiki/Power_set
- http://en.wikipedia.org/wiki/Equivalence_relation
- http://en.wikipedia.org/wiki/Injective_function
- http://en.wikipedia.org/wiki/Surjective_function
- <http://www-history.mcs.st-andrews.ac.uk/Biographies/Smullyan.html> is a biography of the American mathematician, logician and magician **Raymond Merrill Smullyan** (1919–).
- <http://www-history.mcs.st-andrews.ac.uk/Biographies/Boole.html> is a biography of the British mathematician **George Boole** (1815–1864).
- http://www-history.mcs.st-andrews.ac.uk/Biographies/De_Morgan.html is a biography of the British mathematician **Augustus De Morgan** (1806–1871).

Logic Puzzles.

- A **logic puzzle** is a riddle that can be solved by **logical thinking**.

Example (The Island of Knights and Knaves.)

- A certain island has **two types** of inhabitants: knights and knaves.
- **Knights** always tell the truth.
- **Knaves** always lie.
- Every inhabitant is either a knave or a knight.
- You visit the island, and talk to two of its inhabitants, called **A** and **B**.
- **A** says: “Exactly one of us is a knave”.
- **B** says: “At least one of us is a knight.”
- **Who** (if any) **is telling the truth?**

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

Systematical Solution: Table Method.

- For a systematical solution, use a **truth table**.
- On the **left, list** all possible truth values of the claims 'X is a knight' (T for 'true', F for 'false').

A is a knight	B is a knight	Exactly one is a knave	At least one is a knight
T	T	F	T
T	F	T	T
F	T	T	T
F	F	F	F

- On the **right, compute** the corresponding truth values of each of the statements.
- **X** is a knight if and only if **X** speaks the truth. Therefore the entry in the **left** column 'X is a knight' **must be equal** to the **right** entry for **X's** statement.
- Here, row 4 contains the **only match**, hence the **unique solution** of the puzzle.

Further Examples.

- You meet 2 inhabitants of the island.
A: Exactly one of us is a knight.
B: All of us are knaves.
Who (if anyone) is telling the truth?

The following examples illustrate important points.

- You meet 1 inhabitant of the island.
A: I am a knight.

A	A: ...
T	T *
F	F *

(There can be more than one solution.)

- You meet 1 inhabitant of the island.
A: I am a knave.

A	A: ...
T	F
F	T

(No solution? This cannot happen.)

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

A Puzzle With More Than Two Inhabitants.

- You meet 3 inhabitants of the island.

A: Exactly one of us is a knight.

B: All of us are knaves.

C: The other two are lying.

Who (if anyone) is lying?

Solution

A	B	C	A: ...	B: ...	C: ...	
T	T	T	F	F	F	
T	T	F	F	F	F	
T	F	T	F	F	F	
T	F	F	T	F	F	*
F	T	T	F	F	F	
F	T	F	T	F	F	
F	F	T	T	F	T	
F	F	F	F	T	T	

Symbols.

Truth Values

T : **true**

F : **false**

Logical Operations

\wedge : **and** (conjunction)

\vee : **or** (disjunction)

\neg : **not** (negation)

Variables

$a, b, c, \dots, p, q, r, \dots$: any statement

- Let a stand for 'A is a knight' and b for 'B is a knight'.
- Then $\neg a$ means: A is a knave.
- B's statement: 'At least one of us is a knight' (i.e., 'A is a knight' or 'B is a knight') becomes: $a \vee b$.

Note: \vee is an **inclusive** 'or'.

The disjunction $p \vee q$ allows for **both** p and q to be true.

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

Propositional Logic.

- Informally, a **proposition** is a statement that is **unambiguously** either **true** or **false**.
- A **propositional variable** is a symbolic name (like p, q, r, \dots) that stands for an arbitrary proposition.
- Formally, a proposition is defined recursively:

Definition (Formal Proposition)

- Any **propositional variable** is a **formal proposition**.

Moreover, if p and q are formal propositions, the following **compound statements** are **formal propositions**:

- the **conjunction** $p \wedge q$ (read: "p and q"), stating that "both p and q are true";
- the **disjunction** $p \vee q$ (read: "p or q"), stating that "either p or q are true";
- the **negation** $\neg p$ (read: "not p"), stating that "it is not the case that p is true".

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

Truth Tables.

- A **truth table** shows the truth value of a compound statement for every possible combination of truth values of its simple components.

p	q	$p \wedge q$	p	q	$p \vee q$	p	$\neg p$
T	T	T	T	T	T	T	F
T	F	F	T	F	T	F	T
F	T	F	F	T	T	F	T
F	F	F	F	F	F	F	T

Example (The truth table for $(p \vee q) \wedge \neg(p \wedge q)$.)

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \vee q$	$(p \vee q) \wedge \neg(p \wedge q)$
T	T	T	F	T	F
T	F	F	T	T	T
F	T	F	T	T	T
F	F	F	T	F	F

A truth table built from the tables of $p \wedge q$, $p \vee q$ and $\neg p$.

Simplifying Negations.

- In mathematics, propositions often involve formulas.
- The negation of such a proposition can usually be reformulated in simpler terms with different symbols.

Example

- The negation of the statement “ $x < 18$ ” is “ $\neg(x < 18)$ ”, or simply “ $x \geq 18$ ”.
- The negation of a **conjunction** is a **disjunction**(!)

Example (Truth tables for $\neg(p \wedge q)$ and $(\neg p \vee \neg q)$.)

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Logical Equivalence.

- Two statements p and q are **logically equivalent** if they have the **same truth value** for every row of the truth table: We then write $p \equiv q$.

Theorem (DeMorgan's Laws)

Let p and q be propositions. Then

- 1 $\neg(p \vee q) \equiv \neg p \wedge \neg q$;
- 2 $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

- A proposition p is a **tautology**, if its truth value is **T**, for all possible combinations of the truth values of its propositional variables: $p \equiv T$.
- A proposition p is a **contradiction**, if its truth value is **F**, for all possible combinations of the truth values of its propositional variables: $p \equiv F$.
- Every logical equivalence is a tautology.

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

Logical Equivalences.

Theorem (for propositional variables p, q, r .)

All of the following are valid logical equivalences.

- **Commutative Laws:** $p \wedge q \equiv q \wedge p$, and $p \vee q \equiv q \vee p$.
- **Associative Laws:** $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$, and $(p \vee q) \vee r \equiv p \vee (q \vee r)$.
- **Distributive Laws:** $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$, and $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.
- **Absorption Laws:** $p \wedge (p \vee q) \equiv p$, and $p \vee (p \wedge q) \equiv p$.
- **Idempotent Laws:** $p \wedge p \equiv p$, and $p \vee p \equiv p$.
- **Complementary Laws:** $p \wedge \neg p \equiv F$, and $p \vee \neg p \equiv T$.
- **Identity Laws:** $p \wedge T \equiv p$, and $p \vee F \equiv p$.
- **Universal Bound:** $p \wedge F \equiv F$, and $p \vee T \equiv T$.
- **DeMorgan:** $\neg(p \wedge q) \equiv \neg p \vee \neg q$, and $\neg(p \vee q) \equiv \neg p \wedge \neg q$.
- **Negation:** $\neg T \equiv F$, and $\neg F \equiv T$.
- **Double Negation:** $\neg(\neg p) \equiv p$.

Proof: truth tables. □

Sets.

- A **set** is a collection of objects, its **elements**.

Notation.

$a \in S$ means: object a is an element of the set S . And
 $a \notin S$ means: object a is **not** an element of the set S .

- Two sets A and B are **equal** ($A = B$) if they have the same elements:
 $a \in B$ for all $a \in A$ and $b \in A$ for all $b \in B$.

Examples

$\{0, 1\}$,
 $\mathbb{N} = \{1, 2, 3, \dots\}$ (the **natural numbers**),
 $\{x \in \mathbb{N} \mid x \text{ is a multiple of } 5\}$,
 $\emptyset = \{\}$ (the **empty set**).

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

Predicates.

Definition

A **predicate** $P(x)$ is a statement that incorporates a **variable** x , such that whenever x is **replaced by a value**, the resulting statement becomes a **proposition**.

Example

- Suppose $P(n)$ is the **predicate** “ n is even”.
 - Then $P(14)$ is the **proposition** “14 is even”.
 - The proposition $P(13)$ is false.
 - $P(22)$ is true.
- Predicates can be combined using the **logical operators** \wedge (and), \vee (or), \neg (not) to create **compound predicates**.
 - A predicate can have more than one variable, e.g., $P(x, y)$ can stand for the predicate “ $x \leq y$ ”.

Quantified Predicates.

Notation.

- Suppose that $P(x)$ is a predicate and that S is a set.
- “ $\forall a \in S, P(a)$ ” is the proposition: “**for all** elements a of S the statement $P(a)$ is true”.
- “ $\exists a \in S, P(a)$ ” is the proposition: “**there exists** (at least) one element a in the set S such that the statement $P(a)$ is true”.

Suppose $S = \{x_1, x_2, \dots\}$.

- “ $\forall a \in S, P(a)$ ” abbreviates “ $P(x_1) \wedge P(x_2) \wedge \dots$ ”.
- “ $\exists a \in S, P(a)$ ” abbreviates “ $P(x_1) \vee P(x_2) \vee \dots$ ”.

Negating Quantified Predicates.

- The negation of “ $\forall x \in S, P(x)$ ” is “ $\exists x \in S, \neg P(x)$ ”;
- the negation of “ $\exists x \in S, P(x)$ ” is “ $\forall x \in S, \neg P(x)$ ”.

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

Implications.

Definition

An **implication** is a statement of the form “if p then q ”. In symbols, we write this as $p \rightarrow q$ (read: “ p implies q ”). We call proposition p the **hypothesis** and proposition q the **conclusion** of the implication $p \rightarrow q$.

- The **truth table** of $p \rightarrow q$ has the form

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Remark.

The **only way** for an implication $p \rightarrow q$ to be false is when the **hypothesis** p is **true**, but the **conclusion** q is **false**.

Converse, Inverse, Contrapositive.

Various variations of the implication $p \rightarrow q$ are of sufficient interest:

- $q \rightarrow p$ is the **converse** of $p \rightarrow q$.
- $\neg p \rightarrow \neg q$ is the **inverse** of $p \rightarrow q$.
- $\neg q \rightarrow \neg p$ is the **contrapositive** of $p \rightarrow q$.

Remark.

- 1 An implication is logically equivalent to its contrapositive: $p \rightarrow q \equiv \neg q \rightarrow \neg p$.
- 2 The converse and the inverse of an implication are logically equivalent: $q \rightarrow p \equiv \neg p \rightarrow \neg q$.
- 3 But an implication is not logically equivalent to its converse (and hence not to its inverse).

Proof: Truth tables. □

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

Biconditional.

- Write $p \leftrightarrow q$ if both $p \rightarrow q$ and $q \rightarrow p$ are true.
- Then $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$.
- The **truth table** of $p \leftrightarrow q$ has the form

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

- Usually, to prove a statement of the form $p \leftrightarrow q$, one proves the two statements $p \rightarrow q$ and $q \rightarrow p$ separately.

Examples

- n is even if and only if n^2 is even.
- The integer n is a multiple of 10 if and only if it is even.

Validating Arguments.

- An **argument** is a list of **statements**, ending in a **conclusion**.
- The logical **form** of an argument can be abstracted from its **content**.

Definition

Formally, an **argument structure** is a list of statements $p_1, p_2, \dots, p_n, \therefore c$ starting with **premises** p_1, \dots, p_n and ending in a **conclusion** c .

- An argument is **valid** if the conclusion follows **necessarily** from the premises.
- Validity of arguments depends only on the form, not on the content.
- The argument structure ' $p_1, \dots, p_n, \therefore c$ ' is **valid** if the proposition $(p_1 \wedge \dots \wedge p_n) \rightarrow c$ is a **tautology**, otherwise it is **invalid**.

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

How to Test Argument Validity.

- 1 Identify the **premises** and the **conclusion** of the argument.
- 2 Construct a **truth table** showing the truth values of all premises and the conclusion.
- 3 A **critical row** is a row of the truth table in which **all** the **premises** are **true**. Check the critical rows as follows.
- 4 If the **conclusion is true in every critical row** then the **argument structure is valid**.
- 5 If there is a critical row in which the conclusion is false, then it is possible for an argument of the given form to have a **false conclusion despite true premises** and so the **argument structure is invalid**.

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

Example of an Invalid Argument Structure.

Example

- Premises: $p_1 = (p \rightarrow q \vee \neg r)$, $p_2 = (q \rightarrow p \wedge r)$.
- Conclusion: $c = (p \rightarrow r)$.
- The argument structure $p_1, p_2, \therefore c$ is **invalid**:

p	q	r	$\neg r$	$q \vee \neg r$	$p \wedge r$	p_1	p_2	c
T	T	T	F	T	T	T	T	T
T	T	F	T	T	F	T	F	F
T	F	T	F	F	T	F	T	F
T	F	F	T	T	F	T	T	F(!)
F	T	T	F	T	F	T	F	F
F	T	F	T	T	F	T	F	F
F	F	T	F	F	F	T	T	T
F	F	F	T	T	F	T	T	T

Valid Arguments vs Invalid Arguments.

Some Valid Argument Forms.

- Modus ponens: $p \rightarrow q, p, \therefore q$.
- Modus tollens: $p \rightarrow q, \neg q, \therefore \neg p$.
- Generalization: $p, \therefore p \vee q$.
- Specialization: $p \wedge q, \therefore p$.
- Conjunction: $p, q, \therefore p \wedge q$.
- Elimination: $p \vee q, \neg q, \therefore p$.
- Transitivity: $p \rightarrow q, q \rightarrow r, \therefore p \rightarrow r$.
- Division into cases: $p \vee q, p \rightarrow r, q \rightarrow r, \therefore r$.
- Contradiction Rule: $\neg p \rightarrow F, \therefore p$.

Some Common Fallacies.

- Converse fallacy: $p \rightarrow q, q, \therefore p$.
- Inverse fallacy: $p \rightarrow q, \neg p, \therefore \neg q$.

“All Humans Are Mortal.”

- **Modus Ponens:**
 $p \rightarrow q, p, \therefore q$.
- **Modus Tollens:**
 $p \rightarrow q, \neg q, \therefore \neg p$.

Example

- If Socrates is human then he is mortal.
- Socrates is human.
- \therefore Socrates is mortal.

- Proof by truth table:

p	q	$p \rightarrow q$	p	q
T	T	T	T	T
T	F	F	T	
F	T	T	F	
F	F	T	F	

Example

- If Zeus is human then he is mortal.
- Zeus is not mortal.
- \therefore Zeus is not human.

- Proof by truth table:

p	q	$p \rightarrow q$	$\neg q$	$\neg p$
T	T	T	F	
T	F	F	T	
F	T	T	F	
F	F	T	T	T

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

Fallacies.

- **Converse Fallacy:**
 $p \rightarrow q, q, \therefore p$.

Example (WRONG!)

- If Socrates is human then he is mortal.
- Socrates is mortal
- \therefore Socrates is human.

- Truth table:

p	q	$p \rightarrow q$	q	p
T	T	T	T	T
T	F	F	F	
F	T	T	T	F(!)
F	F	T	F	

- **Inverse Fallacy:**
 $p \rightarrow q, \neg p, \therefore \neg q$.

Example (WRONG!)

- If Zeus is human then he is mortal.
- Zeus is not human.
- \therefore Zeus is not mortal.

- Truth table:

p	q	$p \rightarrow q$	$\neg p$	$\neg q$
T	T	T	F	
T	F	F	F	
F	T	T	T	F(!)
F	F	T	T	T

Knights and Knaves Revisited.

- $a =$ 'A is a knight'.
- $b =$ 'B is a knight'.

Example

- You visit the island of knights and knaves and find that:

$$\begin{aligned} a &\rightarrow \neg b \\ \neg a &\rightarrow \neg b \\ b &\rightarrow a \vee b \\ \neg b &\rightarrow \neg a \wedge \neg b \end{aligned}$$

(a 'formal version' of the original puzzle).

- Who (if any) is telling the truth?

Solution

- Start with the tautology $a \vee \neg a$.
- Division into cases:
 $a \vee \neg a,$
 $a \rightarrow \neg b,$
 $\neg a \rightarrow \neg b,$
 $\therefore \neg b.$
- Modus ponens:
 $\neg b \rightarrow \neg a \wedge \neg b,$
 $\neg b,$
 $\therefore \neg a \wedge \neg b.$
- Both are knaves!
- This solution is a 'formal version' of the original solution.

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

Subsets and Set Operations.

- A set B is a **subset** of a set A if each element of B is also an element of A :
 $B \subseteq A$ if $b \in A$ for all $b \in B$.
- $A = B$ if and only if $B \subseteq A$ and $A \subseteq B$.
- We assume that all our sets are subsets of a (big) **universal set**, or **universe** U .

Definition

Let $A, B \subseteq U$.

- The **union** of A and B is the set
 $A \cup B = \{x \in U : x \in A \text{ or } x \in B\}$.
- The **intersection** of A and B is the set
 $A \cap B = \{x \in U : x \in A \text{ and } x \in B\}$.
- The **(set) difference** of A and B is the set
 $A \setminus B = \{x \in U : x \in A \text{ and } x \notin B\}$.
- The **complement** of A (in U) is the set
 $A' = \{x \in U : x \notin A\}$.

Set Equations.

Theorem

Let A, B, C be subsets of a universal set U . Then all of

$$\begin{array}{ll}
 A \cap B = B \cap A, & A \cup B = B \cup A, \\
 (A \cap B) \cap C = A \cap (B \cap C), & (A \cup B) \cup C = A \cup (B \cup C), \\
 A \cap (B \cup C) = (A \cap B) \cup (A \cap C), & A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \\
 A \cap (A \cup B) = A, & A \cup (A \cap B) = A, \\
 A \cap A = A, & A \cup A = A, \\
 A \cap A' = \emptyset, & A \cup A' = U, \\
 A \cap U = A, & A \cup \emptyset = A, \\
 A \cap \emptyset = \emptyset, & A \cup U = U, \\
 (A \cap B)' = A' \cup B', & (A \cup B)' = A' \cap B', \\
 U' = \emptyset, & \emptyset' = U, \\
 (A')' = A &
 \end{array}$$

are valid properties of set operations.

Proof: element-wise. □

MA180-4

The Language of Mathematics: Logic and Sets. Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets. Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

Boolean Algebra.

- An example of **abstraction** in mathematics . . .

- **Sets** (together with the operations \cap , \cup , $'$, and the constants \emptyset , U) behave similar to **Propositions** (together with the operations \wedge , \vee , \neg , and the constants F , T)

- Both are examples of an **abstract structure** (with \cdot , $+$, $'$, and 0 , 1) called a **Boolean algebra**

- For any **logical equivalence**, there is a corresponding **set equality**, and vice versa.

Duality

- The **dual** of a set equality is obtained by swapping \cap with \cup and swapping \emptyset with U .
- The dual of a valid set equality is also a valid set equality . . .

Sets of Sets.

Definition

Let A be a set. The **power set** of A is the set $P(A) = \{B : B \subseteq A\}$ of **all** subsets B of A .

Example

The power set of $A = \{1, 3, 5\}$ is the set $P(A) = \{\emptyset, \{1\}, \{3\}, \{5\}, \{1, 3\}, \{1, 5\}, \{3, 5\}, \{1, 3, 5\}\}$

Definition

A **partition** of a set A is a set $P = \{P_1, P_2, \dots\}$ of **parts** $P_1, P_2, \dots \subseteq A$ such that

- 1 no part is empty: $P_i \neq \emptyset$ for all i ;
- 2 distinct parts are disjoint: $P_i \cap P_j = \emptyset$ for all $i \neq j$;
- 3 every point is in some part: $A = P_1 \cup P_2 \cup \dots$.

MA180-4

The Language of Mathematics: Logic and Sets. Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets. Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

Products of Sets.

Definition

The **Cartesian product** of sets A and B is the set $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$ of all **(ordered) pairs** (a, b) .

Examples

- $A = \{1, 2, 3\}$, $B = \{X, Y\}$.
 $A \times B = \{(1, X), (1, Y), (2, X), (2, Y), (3, X), (3, Y)\}$.
- $A = \{1, 3\}$. $A^2 = A \times A = \{(1, 1), (1, 3), (3, 1), (3, 3)\}$
- More generally, for $n \in \mathbb{N}$, the Cartesian product of n sets S_1, S_2, \dots, S_n is the set $S_1 \times S_2 \times \dots \times S_n = \{(x_1, x_2, \dots, x_n) : x_i \in S_i\}$ of all **n -tuples** (x_1, x_2, \dots, x_n) .
- $A^n = A \times A \times \dots \times A$ (n factors).

Relations are Sets.

- A **relation** from a **domain** X to a **codomain** Y is a subset $R \subseteq X \times Y$.

Notation.

Write xRy (and say “ x is related to y ”) for $(x, y) \in R$.

- Let R be a relation on X , i.e, $R \subseteq X \times X$.
- R is **reflexive** if xRx for all $x \in X$.
- R is **symmetric** if xRy then yRx for all $x, y \in X$.
- R is **transitive** if xRy and yRz then xRz , for all $x, y, z \in X$.
- A relation $R \subseteq X \times X$ that is reflexive, symmetric and transitive is called an **equivalence relation**.

MA180-4

The Language of Mathematics: Logic and Sets.
Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.
Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.
Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets.
Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.
Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.
Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

Equivalence Relations are Partitions.

- Suppose R is an **equivalence relation** on a set X . For $x \in X$, denote by $[x] = \{y : xRy\}$ the **equivalence class** of x , i.e., the set of all $y \in X$ that x is R -related to. Also denote by $X/R = \{[x] : x \in X\}$ the **quotient set**, i.e., the set of all equivalence classes.
- Suppose that P is a **partition** of X . For $x \in X$, denote by $P(x)$ the **unique part** of P that contains x .

Theorem

- 1 If R is an **equivalence relation** on the set X , then the quotient set X/R is a **partition** of X .
- 2 Conversely, if P is a **partition** of a set X , then the relation $R = \{(x, y) \in X^2 : P(x) = P(y)\}$ is an **equivalence relation**.

Functions are Relations are Sets.

- A **function** f from a **domain** X to a **codomain** Y is a **relation** $f \subseteq X \times Y$, with the property that, for every $x \in X$, there is a **unique** $y \in Y$ such that $(x, y) \in f$.
- (This is often called the **Vertical Line Test**.)

Notation.

Write $f: X \rightarrow Y$ for a function f from X to Y and $f(x) = y$ for the unique $y \in Y$ such that if $(x, y) \in f$.

- A **function** thus consists of three things: a **domain** X and a **codomain** Y together with a **rule** $f \subseteq X \times Y$ that associates to each point $x \in X$ a **unique value** $f(x) = y \in Y$.

MA180-4

The Language of Mathematics: Logic and Sets.
Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.
Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.
Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets.
Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.
Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.
Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

Injective and Surjective Functions.

- A function $f: X \rightarrow Y$ is called **surjective** (or **onto**) if,

for every $y \in Y$,
there is **at least** one $x \in X$ such that $f(x) = y$.

- A function $f: X \rightarrow Y$ is called **injective** (or **one-to-one**) if,

for every $y \in Y$,
there is **at most** one $x \in X$ such that $f(x) = y$.

- A function $f: X \rightarrow Y$ is called **bijective** (or a **one-to-one correspondence** if it is both **injective** and **surjective**, i.e., if,

for every $y \in Y$,
there is a **unique** $x \in X$ such that $f(x) = y$.

- A function is injective/surjective/bijective if it passes a suitable **Horizontal Line Test**.

Bijections of Partitions and Subsets.

- Consider a **function** $f: X \rightarrow Y$.
- The **image** $f(X) = \{f(x) : x \in X\}$ is a **subset** of Y .
- The **relation** \sim_f on X by $x \sim_f x'$ if $f(x) = f(x')$ is an **equivalence** relation and the equivalence classes $[x] = \{x' \in X : f(x) = f(x')\}$ form **partition** X/\sim_f of X , called the **kernel** of f .

Theorem

- 1 *Let $f: X \rightarrow Y$. Then the function $F: X/\sim_f \rightarrow f(X)$ defined by $F([x]) = f(x)$ for $x \in X$ is a well-defined bijection between the kernel X/\sim_f of f and the image $f(X)$ of f .*
- 2 *Conversely, if $Y' \subseteq Y$ is any subset of Y , if \sim is any equivalence relation on X and $F: X/\sim \rightarrow Y'$ is a bijection then the rule $f(x) = F([x])$ defines a function f from X to Y .*

MA180-4

The Language of Mathematics: Logic and Sets. Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials. Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets. Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials. Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

Summary: The Language of Mathematics.

- **Formal propositions** consist of **propositional variables**, combined by the **logical connectives** \wedge (and), \vee (or), and \neg (not).
- A **truth table** determines the truth value of a proposition depending on the truth values of its propositional variables.
- Truth tables can **validate** and invalidate **argument structures**.
- Sets, with the operations \cap (intersection), \cup (union), and $'$ (complement in a universal set U) form a **Boolean algebra**, like the formal propositions with their logical operations.
- Claims about sets are **proved** by valid arguments.
- **Functions** and **relations** are sets (of pairs).
- A function is a one-to-one **correspondence** between a **partition** of its **domain** and a **subset** of its **codomain**.

Introduction: Permutations and Polynomials.

- Certain **types of functions** occur frequently in **applications** and form examples of important **algebraic structures**.
- **Permutations** of a set correspond to **rearrangements** of its elements.
- In Computer Science, permutations are used in the study of **sorting** algorithms.
- The **product** of two permutations is a **composition** of functions.
- **Polynomials** are linear combinations of powers of an **indeterminate** x .
- Solving **polynomial equations** is a central problem in algebra.
- **Addition**, **multiplication** and **division** of polynomials share many properties with the corresponding operations on the **integers**.

MA180-4

The Language of Mathematics: Logic and Sets. Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials. Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets. Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials. Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

Links: Permutations and Polynomials.

- http://en.wikipedia.org/wiki/Fifteen_puzzle
- http://en.wikipedia.org/wiki/Rubik's_Cube
- http://en.wikipedia.org/wiki/Function_composition
- <http://en.wikipedia.org/wiki/Permutation>
- http://en.wikipedia.org/wiki/Symmetric_group
- [http://en.wikipedia.org/wiki/Cycle_\(mathematics\)](http://en.wikipedia.org/wiki/Cycle_(mathematics))
- <http://www.cut-the-knot.org/Curriculum/Combinatorics/PermByTrans.shtml>
- [http://en.wikipedia.org/wiki/Group_\(mathematics\)](http://en.wikipedia.org/wiki/Group_(mathematics))
- [http://en.wikipedia.org/wiki/Ring_\(mathematics\)](http://en.wikipedia.org/wiki/Ring_(mathematics))
- [http://en.wikipedia.org/wiki/Field_\(mathematics\)](http://en.wikipedia.org/wiki/Field_(mathematics))
- <http://en.wikipedia.org/wiki/Polynomial>
- http://en.wikipedia.org/wiki/Polynomial_long_division
- http://en.wikipedia.org/wiki/Irreducible_polynomial
- <http://mathworld.wolfram.com/IrreduciblePolynomial.html>
- http://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra

Composition of Functions.

- The **composition** of relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is the relation $S \circ R$ from X to Z defined by $x(S \circ R)z$ if xRy and ySz for some $y \in Y$.
- The **composition** of functions $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ is the function $g \circ f: X \rightarrow Z$ defined by $(g \circ f)(x) = g(f(x))$ for $x \in X$.

Theorem

*Composition of functions is **associative**:*
 $(f \circ g) \circ h = f \circ (g \circ h)$.

Proof.

$$((f \circ g) \circ h)(x) = (f \circ g)(h(x)) = f(g(h(x))) \\ = f((g \circ h)(x)) = (f \circ (g \circ h))(x). \quad \square$$

- The composition of functions $f: X \rightarrow X$ and $g: X \rightarrow X$ is a function $g \circ f$ from the set X to itself.

MA180-4

The Language of Mathematics: Logic and Sets. Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets. Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

Bijections and Inverse functions.

Example

Let X be a set. The **identity function** $\text{id}_X: X \rightarrow X$, defined by $\text{id}_X(x) = x$ for all $x \in X$, is a bijection.

- If $f: X \rightarrow Y$ is a bijection there is a function $g: Y \rightarrow X$ defined by $g(y) = x$ if $f(x) = y$ (i.e., g maps $y \in Y$ to the unique $x \in X$ that f maps to y .)
- The function g is bijective as well and has the property that $g \circ f = \text{id}_X$ (i.e., $g(f(x)) = x$ for all $x \in X$) and $f \circ g = \text{id}_Y$ (i.e. $f(g(y)) = y$ for all $y \in Y$).
- This function g is **uniquely determined** by f and called the **inverse** of f .

Theorem

*A function has an **inverse** if and only if it is a **bijection**.*

Permutations.

- A **permutation** of a set X is a **bijection** from X to **itself**.
- Frequently, $X = \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$.

Example ($X = \{1, 2, 3, 4, 5, 6\}$.)

The **relation** $\pi = \{(1, 2), (2, 5), (3, 3), (4, 6), (5, 1), (6, 4)\}$ on X is a **bijection**, which is written in **two-line-notation** as the **permutation** $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 3 & 6 & 1 & 4 \end{pmatrix}$.

- There are $n! = 1 \cdot 2 \cdots n$ permutations of X if $|X| = n$.
- The set S_n of all permutations of $X = \{1, 2, \dots, n\}$ is called the **symmetric group** of degree n .

Example ($n = 3$.)

$S_3 = \{(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{smallmatrix}), (\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{smallmatrix}), (\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{smallmatrix}), (\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix}), (\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{smallmatrix}), (\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{smallmatrix})\}$. $|S_3| = 3! = 6$.

MA180-4

The Language of Mathematics: Logic and Sets. Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets. Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

Products of Permutations.

- The **product** $\sigma \circ \pi$ of $\pi, \sigma \in S_n$, defined by $(\sigma \circ \pi)(x) = \sigma(\pi(x))$, for $x \in X$, is a permutation.
- The **inverse** of $\pi = \begin{pmatrix} 1 & \dots & n \\ \pi(1) & \dots & \pi(n) \end{pmatrix}$ is the permutation $\pi^{-1} = \begin{pmatrix} \pi(1) & \dots & \pi(n) \\ 1 & \dots & n \end{pmatrix}$, since $\pi^{-1} \circ \pi = \text{id}_X$.
- An **m-cycle** (x_1, x_2, \dots, x_m) permutes the **m** points $x_1, x_2, \dots, x_m \in X$ cyclically.
- Each permutation is a product of **disjoint cycles**.

Example

$$\pi = \begin{pmatrix} 123456 \\ 253614 \end{pmatrix} = (1, 2, 5)(3)(4, 6) = (1, 2, 5)(4, 6).$$

- The **order** of a permutation π is the smallest $k \in \mathbb{N}$ such that $\pi^k = \pi \circ \pi \circ \dots \circ \pi = \text{id}_X$.
- An **m-cycle** has order **m**.
- The order of $\pi \in S_n$ is the **lcm** of its cycle lengths.

Write a Permutation as Disjoint Cycles.

Algorithm: Disjoint Cycles.

0. Consider all points $x \in \{1, 2, \dots, n\}$ as “**unmarked**”.
1. If all points are marked: STOP
Otherwise, let x be the **smallest unmarked point**.
2. Determine its **cycle**

$$(x, \pi(x), \pi^2(x), \dots)$$

and mark all the points in the cycle.

3. **Go back** to step 1.

- Here $\pi^2 = \pi \circ \pi$, $\pi^k = \pi \circ \pi^{k-1}$.
- Given $\pi \in S_n$, what is the smallest $k \in \mathbb{N}$, such that $\pi^k = \text{id}_X$?
- This **k** is called the **order** of π .

MA180-4

The Language of Mathematics: Logic and Sets.
Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets.
Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

Products of Transpositions.

Examples

$$(1, 2)^{-1} = (1, 2) \text{ and } (1, 2)(2, 3) = (1, 2, 3).$$

- A **2-cycle** is called a **transposition**.
- Each **n-cycle** is a product of transpositions:
 $(x_1, x_2, \dots, x_n) = (x_1, x_2)(x_2, x_3) \cdots (x_{n-1}, x_n).$

Theorem (Librarian's Nightmare.)

Each permutation $\pi \in S_n$ is a product of transpositions

- $\pi \in S_n$ is called **even** (resp. **odd**) if it is a product of an even (resp. odd) number of transpositions.

Fact.

A permutation $\pi \in S_n$ is either even or odd, **but not both**.

Groups.

- The symmetric group S_n is an example of a **group**.
- In general, a group is defined by **axioms**.

Definition

A **group** is a set G , together with a **binary operation**

$\star: G \times G \rightarrow G$ such that:

- (G1) **Associative:** $(a \star b) \star c = a \star (b \star c)$ for all $a, b, c \in G$.
 (G2) **Identity:** There exists an element $e \in G$ such that $a \star e = a$ and $e \star a = a$ for all $a \in G$.
 (G3) **Inverse:** For each $a \in G$ there exists an element $a' \in G$ such that $a \star a' = e$ and $a' \star a = e$.

Examples

- $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, (\mathbb{Q}^*, \cdot) , $(\mathbb{Z}_n, +)$, (\mathbb{Z}_n^*, \cdot) , $(\{\pm 1\}, \cdot)$, ...
- The set of invertible 2×2 -matrices over \mathbb{Q} .
- $(\mathbb{N}, +)$, (\mathbb{Z}, \cdot) , $(P(S), \cup)$, $(P(S), \cap)$ are **not groups**.

MA180-4

The Language of Mathematics: Logic and Sets.
Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets.
Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

Rings.

- A group (G, \star) is **abelian** (or **commutative**) if $a \star b = b \star a$ for all $a, b \in G$.

Definition

A **ring** is a set R together with binary operations $+$ and $\star: R \times R \rightarrow R$ such that $(R, +)$ is an abelian group and:

$$(R1) \quad (a \star b) \star c = a \star (b \star c) \text{ for all } a, b, c \in R.$$

$$(R2) \quad \text{There exists an element } e \in R \text{ such that } a \star e = a \text{ and } e \star a = a \text{ for all } a \in R.$$

$$(R3) \quad a \star (b + c) = a \star b + a \star c \text{ and } (a + b) \star c = a \star c + b \star c \text{ for all } a, b, c \in R.$$

- A ring $(R, +, \star)$ is called **commutative** if $a \star b = b \star a$ for all $a, b \in R$.

Examples

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}_m, \dots$, the set of **all** 2×2 -matrices over \mathbb{Q} .

Polynomials are like Numbers.

Definition

Suppose R is a commutative ring. A **polynomial** over R is an expression of the form

$$\begin{aligned} p(x) &= a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \\ &= a_0 + a_1 x + \dots + a_{n-1} x^{n-1} + a_n x^n = \sum_{i=0}^n a_i x^i. \end{aligned}$$

for some integer $n \geq 0$, with **coefficients** $a_0, a_1, \dots, a_n \in R$ (e.g. $R = \mathbb{R}$ or $R = \mathbb{Z}_m$)

- Two polynomials are **equal** if they have the **same coefficient** at every power of x .
- A polynomial $p(x)$ defines a **polynomial function** $R \rightarrow R$ by the rule $a \mapsto p(a)$.

Distinct polynomials can define the same function.

MA180-4

The Language of Mathematics: Logic and Sets. Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions. Permutations. **Polynomials.** Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets. Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions. Permutations. **Polynomials.** Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

Rings of Polynomials.

- The set of all polynomials over R is denoted by $R[x]$.
- Polynomials can be **added**:

$$\left(\sum a_i x^i \right) + \left(\sum b_i x^i \right) = \sum (a_i + b_i) x^i.$$

- Polynomials can be **multiplied**:

$$\begin{aligned} \left(\sum a_i x^i \right) \left(\sum b_i x^i \right) &= \sum_j \sum_k a_j b_k x^{j+k} \\ &= \sum_i \left(\sum_{j+k=i} a_j b_k \right) x^i. \end{aligned}$$

- $R[x]$ is a **commutative ring**.

Quotients and Roots of Polynomials.

- A **field** is a commutative ring F , where each $a \in F \setminus \{0\}$ has an inverse.

Examples

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are fields, \mathbb{Z} is not. \mathbb{Z}_m is a field if m is a prime.

Theorem

Suppose that F is a field.

- (**Division Theorem.**) Let $f, g \in F[x]$ be polynomials with $g \neq 0$. Then there exist unique polynomials $q \in F[x]$ (the **quotient**) and $r \in F[x]$ (the **remainder**) with $\deg r < \deg g$ such that $f = gq + r$.
- (**Remainder Theorem.**) For any polynomial $f(x) \in F[x]$ and $a \in F$, the value $f(a)$ is the remainder of $f(x)$ upon division by $(x - a)$.
- (**Root Theorem.**) $a \in F$ is a **root** of $f(x) \in F[x]$ if and only if $x - a$ is a **factor** of $f(x)$.

MA180-4

The Language of Mathematics: Logic and Sets. Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions. Permutations. **Polynomials.** Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets. Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions. Permutations. **Polynomials.** Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction. Examples and Applications of Induction. Determinants. Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

Greatest Common Divisors.

- **Euclid's Algorithm** can be used to compute the **gcd** of two polynomials f and g .

Example

- $f = x^5 + 1 \in \mathbb{Z}_3[x]$, $g = x^2 + 1 \in \mathbb{Z}_3[x]$.
- $x^5 + 1 = (x^2 + 1)(x^3 + 2x) + (x + 1)$.
- $x^2 + 1 = (x + 1)(x + 2) + 2$.
- $\gcd(f, g) = 2 = -1 \cdot 1$.

Example

- $f = x^3 + 2x^2 + 2 \in \mathbb{Z}_3[x]$, $g = x^2 + 2x + 1 \in \mathbb{Z}_3[x]$.
- $x^3 + 2x^2 + 2 = (x^2 + 2x + 1)x + (2x + 2)$.
- $x^2 + 2x + 1 = (2x + 2)(2x + 2) + 0$.
- $\gcd(f, g) = 2x + 2 = -1 \cdot (x + 1)$.
- $\gcd(f, g)$ can be computed without factoring f or g .

Irreducible Polynomials.

- Recall that, if $f \in F[x]$ then $\deg f \leq 0$ if and only if f is a constant polynomial, i.e. $f \in F$.
- A polynomial $p \in F[x]$ is **irreducible** if $\deg p > 0$ and if $p = fg$ for polynomials $f, g \in F[x]$ implies that either $\deg f = 0$ or $\deg g = 0$.
- Any nonzero polynomial $f \in F[x]$ is either irreducible or it is a **product** of **irreducible polynomials**.

Theorem

Let $f \in F[x]$. If $f = p_1 p_2 \cdots p_s$ and $f = q_1 q_2 \cdots q_t$ are two factorizations of f into a product of irreducible polynomials, then $s = t$, and up to rearranging the factors, $q_i = r_i p_i$ for some $r_i \in F$, $i = 1, \dots, s$.

- Thus the factorization of a polynomial f into a product of irreducible polynomials is essentially **unique**.

MA180-4

The Language of Mathematics: Logic and Sets.
Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.
Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.
Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets.
Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.
Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.
Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

Examples of Irreducible Polynomials.

- $f(x) = x - r \in F[x]$ for any $r \in F$ is irreducible.
- $f(x) = x^2 + bx + c \in \mathbb{R}[x]$ is irreducible if $b^2 - 4c < 0$.

Theorem (Fundamental Theorem of Algebra)

If $f(x) \in \mathbb{C}[x]$ is a polynomial of degree $n > 0$ then $f(x)$ has a root in \mathbb{C} .

- Consequently, no polynomial $f \in \mathbb{C}[x]$ with $\deg f > 1$ is irreducible.
- No polynomial $f \in \mathbb{R}[x]$ with $\deg f > 2$ is irreducible.

Proof.

Suppose $\deg f > 2$. By the Fundamental Theorem, $f(x)$ has a complex root $\alpha \in \mathbb{C}$. Note that $\overline{f(x)} = f(\bar{x})$. $f(\alpha) = 0$ implies $f(\bar{\alpha}) = \overline{f(\alpha)} = \bar{0} = 0$. Hence both $(x - \alpha)$ and $(x - \bar{\alpha})$ are factors of $f(x)$. Suppose $\alpha = a + bi$. Then $(x - \alpha)(x - \bar{\alpha}) = x^2 - 2ax + (a^2 + b^2) \in \mathbb{R}[x]$ is an irreducible factor of $f(x)$. \square

Summary: Permutations and Polynomials.

- **Composition** of functions is **associative**.
- A **permutation** is a bijection from a set to itself.
- A permutation is a product of **disjoint cycles**.
- The cycle lengths determine the **order** of a permutation.
- A permutation has **sign** $(-1)^\ell$ if it is a product of ℓ **transpositions**.
- The permutations of the set $\{1, \dots, n\}$ form the **symmetric group** S_n with **composition** as product.
- The **polynomials** over a **commutative ring** R form a **commutative ring** $R[x]$.
- **Quotients** and **remainders** of polynomials are computed by **long division**.
- A polynomial over a **field** is a product of **irreducible** polynomials in an essentially **unique** way.
- Every irreducible polynomial $f \in \mathbb{C}[x]$ has degree **1**.
- An irreducible polynomial $f \in \mathbb{R}[x]$ has $\deg f \leq 2$.

MA180-4

The Language of Mathematics: Logic and Sets.
Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.
Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.
Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets.
Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.
Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.
Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

Introduction: Induction and Matrix Algebra

- Induction, in the **experimental sciences**, is a type of reasoning used to infer an event from the observation of past events.
- Mathematics is an **exact science**, where this type of reasoning is not considered valid.
- **Mathematical Induction** is a technique used to prove statements about natural numbers.
- Here, properties of the numbers $1, \dots, n - 1$ are used to **prove** a property of the number n .
- The technique **applies** to theorems about **numbers**, about **polynomials**, about **matrices**, ...
- Insights into properties of **square matrices** are obtained by computing their **determinants** and **eigenvalues**.
- Eigenvalues can be found as **roots** of the **characteristic polynomial** of a matrix.

Links: Induction and Matrix Algebra.

- <http://oeis.org/search?q=2,4,8,16,31>
- http://en.wikipedia.org/wiki/Mathematical_induction
- <http://www.cut-the-knot.org/induction.shtml>
- <http://en.wikipedia.org/wiki/Determinant>
- <http://mathworld.wolfram.com/Determinant.html>
- http://en.wikipedia.org/wiki/Adjugate_matrix
- [http://en.wikipedia.org/wiki/Minor_\(linear_algebra\)](http://en.wikipedia.org/wiki/Minor_(linear_algebra))
- http://en.wikipedia.org/wiki/Eigenvalues_and_eigenvectors
- <http://mathworld.wolfram.com/Eigenvalue.html>
- http://en.wikipedia.org/wiki/Characteristic_polynomial
- http://en.wikipedia.org/wiki/Cayley-Hamilton_theorem
- <http://www-history.mcs.st-andrews.ac.uk/Biographies/Cayley.html> is a biography of the British mathematician **Arthur Cayley** (1821–1895).
- <http://www-history.mcs.st-andrews.ac.uk/Biographies/Hamilton.html> is a biography of the Irish mathematician **William Rowan Hamilton** (1805–1865).

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

The Principle of Induction.

- A **statement about the natural numbers** is a **predicate** $P(n)$ with **domain** \mathbb{N} .
- That is, when any natural number is substituted for n then $P(n)$ becomes a **proposition**, a statement that is unambiguously true or false.

Principle of Mathematical Induction

Let $P(n)$ be a statement about the natural numbers. If

- 1 $P(1)$ is true, and
 - 2 $P(k)$ implies $P(k + 1)$, for every integer $k > 0$,
- then we can conclude that
- $P(n)$
- is true for every
- $n \in \mathbb{N}$
- .

- $P(1)$ is called the **base case**.
- A proof that $P(k)$ implies $P(k + 1)$ for all $k > 0$ is called the **induction step**.

An Example of Mathematical Induction.

- Suppose that $a_n = \begin{cases} 1, & \text{if } n = 1, \\ a_{n-1} + (2n - 1), & \text{if } n > 1. \end{cases}$

- **Claim:** $a_n = n^2$ for all $n > 0$.

Proof.

Let $P(n)$ be the statement " $a_n = n^2$ ".

Base Case. $P(1)$ is the statement " $a_1 = 1^2$ ".
 $P(1)$ is true since both $a_1 = 1$ and $1^2 = 1$.

Induction Step. Let $k > 0$.

Assume that $P(k)$ is true, i.e., that $a_k = k^2$.
 $P(k + 1)$ is the statement " $a_{k+1} = (k + 1)^2$ ".

By definition $a_{k+1} = a_k + 2k + 1$.

Using $P(k)$, conclude that $a_{k+1} = k^2 + 2k + 1$.

Now $P(k + 1)$ is true since $(k + 1)^2 = k^2 + 2k + 1$.

Consequently, $P(n)$ is true for all $n > 0$. □

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

Variations of the Induction Theme.

- Sometimes the base case is different from $k = 1$.

Let $P(n)$ be a statement about the integers. If

- 1 $P(1)$ is true for some $1 \in \mathbb{Z}$, and
- 2 $P(k)$ implies $P(k + 1)$, for every integer $k \geq 1$,

then $P(n)$ is true for every integer $n \geq 1$.

- Sometimes it is necessary to assume $P(m)$ for **all** $m \leq k$ in order to derive $P(k + 1)$.

Let $P(n)$ be a statement about the integers. If

- 1 $P(1)$ is true, and
- 2 for every integer $k \geq 1$, the truth of $P(m)$ for all $1 \leq m \leq k$ implies $P(k + 1)$,

then $P(n)$ is true for every integer $n \geq 1$.

Applications of Mathematical Induction.

Counting Subsets.

- A set X of size n has exactly 2^n subsets:
 $|X| = n \implies |P(X)| = 2^n$.

Sums of Integers, Squares, Cubes, ...

- $1 + 2 + \dots + n = \frac{1}{2}n(n + 1)$ for all $n \in \mathbb{N}$.
- $1^2 + 2^2 + \dots + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$ for all $n \in \mathbb{N}$.
- $1^3 + 2^3 + \dots + n^3 = (1 + 2 + \dots + n)^2$ for all $n \in \mathbb{N}$.

Each Permutation is a Product of Transpositions.

- $(x_1, x_2, \dots, x_n) = (x_1, x_2)(x_2, x_3) \dots (x_{n-1}, x_n)$.

The Roots of a Complex Polynomial.

- If $f(x) \in \mathbb{C}[x]$ is a polynomial of degree d , then $f(x)$ has exactly d (not necessarily distinct) roots.

MA180-4

The Language of Mathematics: Logic and Sets.
Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.
Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.
Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets.
Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.
Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.
Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

Determinants.

- Recall $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.
- And $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - afh - bdi - ceg$.

- In general, if $A = (a_{ij})$ is an $n \times n$ -matrix then the **determinant** of A is the **number**

$$\det(A) = |A| = \sum_{\pi \in S_n} \text{sign}(\pi) a_{1,\pi(1)} a_{2,\pi(2)} \dots a_{n,\pi(n)},$$

a sum of $n!$ terms.

- This formula is used for **theoretical** purposes.

Properties of the Determinant.

- $\det(A^T) = \det(A)$, where A^T is the **transpose** of A .
- $\det(AB) = \det(A) \det(B)$, if A and B are both $n \times n$,
- $\det(A) \neq 0$ if and only if A is invertible.

More about Determinants.

Further Properties of the Determinant.

- $\det(I_n) = 1$, where I_n is the $n \times n$ **identity** matrix.
- $\det(A) = 0$ if two rows of A are the **same**.
- $\det(A)$ is **linear** in the i th row of A , for each i .

The Determinant under Row Operations.

- If B is obtained from A by adding a multiple of row i to row j , ($j \neq i$) then $\det B = \det A$.
- If B is obtained from A by multiplying row i with a scalar c then $\det B = c \det A$.
- If B is obtained from A by swapping rows i and j ($j \neq i$) then $\det B = -\det A$.

MA180-4

The Language of Mathematics: Logic and Sets.
Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.
Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.
Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets.
Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.
Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.
Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

Cofactors.

- The **minor matrix** A_{ij} is obtained from $A = (a_{ij})$ by deleting its i th row and its j th column:

$$A_{ij} = \begin{pmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & & & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{pmatrix}$$

- The **cofactor** a'_{ij} of a_{ij} in A is the number $a'_{ij} = (-1)^{i+j} |A_{ij}|$.

$$|A| = a_{11}a'_{11} + a_{12}a'_{12} + \dots + a_{1n}a'_{1n}.$$

Example Determinant Calculation.

- $A = \begin{bmatrix} 5 & -2 & 4 & -1 \\ 0 & 1 & 5 & 2 \\ 1 & 2 & 0 & 1 \\ -3 & 1 & -1 & 1 \end{bmatrix}$.

- $|A| = 5|A_{11}| - (-2)|A_{12}| + 4|A_{13}| - (-1)|A_{14}|$, where

- $|A_{11}| = \begin{vmatrix} 1 & 5 & 2 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} - 5 \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + 2 \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} = 1 - 5 - 4 = -8$,

- $|A_{12}| = \begin{vmatrix} 0 & 5 & 2 \\ 1 & 0 & 1 \\ -3 & -1 & 1 \end{vmatrix} = 0 - 5 \begin{vmatrix} 1 & 1 \\ -3 & 1 \end{vmatrix} + 2 \begin{vmatrix} 1 & 0 \\ -3 & -1 \end{vmatrix} = -20 - 2 = -22$,

- $|A_{13}| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ -3 & 1 & 1 \end{vmatrix} = 0 - 1 \begin{vmatrix} 1 & 1 \\ -3 & 1 \end{vmatrix} + 2 \begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix} = -4 + 14 = 10$,

- $|A_{14}| = \begin{vmatrix} 0 & 1 & 5 \\ 1 & 2 & 0 \\ -3 & 1 & -1 \end{vmatrix} = 0 - 1 \begin{vmatrix} 1 & 0 \\ -3 & -1 \end{vmatrix} + 5 \begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix} = 1 + 35 = 36$.

- So $|A| = 5 \cdot (-8) + 2 \cdot (-22) + 4 \cdot 10 + 36 = \underline{-8}$.

MA180-4

The Language of Mathematics: Logic and Sets. Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials. Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra. Mathematical Induction. Examples and Applications of Induction. **Determinants.** Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets. Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials. Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra. Mathematical Induction. Examples and Applications of Induction. **Determinants.** Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

The Adjoint of a Matrix.

- For a square matrix $A = (a_{ij})$ let $A' = (a'_{ij})$ be the **matrix of cofactors** $a'_{ij} = (-1)^{i+j} |A_{ij}|$ of A .
- The **adjoint matrix** A^* of A is the **transpose** of A' : $A^* = (A')^T$.

Properties

- $A^* \cdot A = A \cdot A^* = |A| \cdot I_n$, where I_n is the $n \times n$ identity matrix.
- Expansion** by the r th **row**: $|A| = a_{r1}a'_{r1} + a_{r2}a'_{r2} + \dots + a_{rn}a'_{rn}$ for each row index r
- Expansion** by the s th **column**: $|A| = a_{1s}a'_{1s} + a_{2s}a'_{2s} + \dots + a_{ns}a'_{ns}$ for each column index s

MA180-4

The Language of Mathematics: Logic and Sets. Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials. Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra. Mathematical Induction. Examples and Applications of Induction. **Determinants.** Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets. Propositional Logic. Valid Arguments. Sets and Boolean Algebra. Functions and Relations. Summary.

Examples of Algebraic Objects: Permutations and Polynomials. Composition of Functions. Permutations. Polynomials. Factorisation of Polynomials. Summary.

Mathematical Tools: Induction and Matrix Algebra. Mathematical Induction. Examples and Applications of Induction. **Determinants.** Eigenvalues and Eigenvectors. Summary.

Course Summary and Outlook.

Upper Triangular Matrices.

- A square matrix $U = (u_{ij})$ is called an **upper triangular matrix** if $u_{ij} = 0$ whenever $i < j$, i.e., if it has the form

$$U = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u_{nn} \end{pmatrix}$$

- The **determinant** of an upper triangular matrix U is the **product** of its **diagonal entries**:

$$|U| = u_{11}u_{22} \dots u_{nn}.$$

Proof.

by induction on n . □

- A similar result holds for **lower triangular matrices**.

Eigenvalues and Eigenvectors.

- Any $n \times n$ -matrix A can be regarded as a **linear transformation** on the vector space \mathbb{R}^n which maps a (column) vector $v \in \mathbb{R}^n$ to the (column) vector $Av \in \mathbb{R}^n$.

- A number λ is an **eigenvalue** of A if there is a **nonzero** vector $v \in \mathbb{R}^n$ such that $Av = \lambda v$.
- The vector v is then called an **eigenvector** of A for the eigenvalue λ .

Example

- $\begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- Here the eigenvalue is $\lambda = 4$ and $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector.

Computing Eigenvalues.

- Write $Av = \lambda v$ as $Av - \lambda v = 0$, or $(A - \lambda I_n)v = 0$, where I_n is the **identity matrix**.
- A number λ is an eigenvalue of A if and only if the system $(A - \lambda I_n)v = 0$ of linear equations has a **nontrivial solution** v .
- $(A - \lambda I_n)v = 0$ has a nontrivial solution v if and only if $\det(A - \lambda I_n) = 0$.

Definition

The **polynomial** $f_A(x) = \det(A - xI_n)$ is the **characteristic polynomial** of the matrix A .

Theorem

A number λ is an **eigenvalue** of the matrix A if and only if λ is a **root of the characteristic polynomial** $f_A(x)$, i.e., $f_A(\lambda) = 0$.

MA180-4

The Language of Mathematics:
Logic and Sets.
Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects:
Permutations and Polynomials.
Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.

Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics:
Logic and Sets.
Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects:
Permutations and Polynomials.
Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.

Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

Computing Eigenvectors; Diagonalization.

- To **find an eigenvector** v for an eigenvalue λ solve the system of linear equations $(A - \lambda I_n)v = 0$ for v and find a **nontrivial solution**.
- Let E be the matrix which has as its **columns** eigenvectors v_1, \dots, v_n corresponding to the eigenvalues $\lambda_1, \dots, \lambda_n$ of A .
- Let D be the **diagonal matrix** with the eigenvalues $\lambda_1, \dots, \lambda_n$ on its diagonal (and all other entries 0).
- Then $AE = ED$.
- If the eigenvalues are **distinct** then E is **invertible** and $A = EDE^{-1}$.

Example

$$\begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}^{-1}$$

Properties of Eigenvalues and Eigenvectors.

Suppose A is an $n \times n$ matrix.

- $\det A$ equals the **product** of the eigenvalues; and **trace** $A = a_{11} + a_{22} + \dots + a_{nn}$ equals their **sum**.

If λ is an eigenvalue of A with eigenvector v then

- λ^k is an eigenvalue of the k th **power** A^k with the same eigenvector v .
- if the **inverse** A^{-1} exists, λ^{-1} is an eigenvalue of A^{-1} with the same eigenvector v .
- $\lambda + \mu$ is an eigenvalue of $A + \mu I_n$ with the same eigenvector v .

Moreover,

- A and its **transpose** A^T have the same eigenvalues;
- if P is an invertible matrix then $P^{-1}AP$ and A have the same eigenvalues.

MA180-4

The Language of Mathematics:
Logic and Sets.
Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects:
Permutations and Polynomials.
Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.

Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics:
Logic and Sets.
Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects:
Permutations and Polynomials.
Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.

Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

Summary: Induction and Matrix Algebra.

- Mathematical **induction** is a powerful tool to prove statements about the natural numbers.
- A proof by mathematical induction consists of 1. the explicit verification of a **base case**, 2. an **induction step** that derives the next case from previous cases.
- An **eigenvalue** of a square matrix A is a number λ such that $Av = \lambda v$ for some vector $v \neq 0$.
- An **eigenvector** of a square matrix A is a vector $v \neq 0$ such that $Av = \lambda v$ for some number λ .
- The eigenvalues of A are the **roots** of the **characteristic polynomial** $\det(A - \lambda I_n)$ of A .
- An eigenvector for the eigenvalue λ is a **nontrivial solution** x of the system $(A - \lambda I_n)x = 0$.
- If the eigenvalues of A are **distinct**, the corresponding eigenvectors form an invertible matrix E which **diagonalizes** A as $A = EDE^{-1}$.

Course Summary and Outlook.

- **Logic** and **Set Theory** form the basis of the language of mathematics.
- Properties of **functions** and **relations** are studied in **Discrete Mathematics (MA284)**.
- **Permutations** are examples of **group elements**.
- **Groups** are studied in **Group Theory: (MA3343, MA4344)**.
- **Polynomials** (with coefficients from a **field**) and **matrices** are examples of **ring elements**.
- **Rings** and **Fields** are studied in **Abstract Algebra: (MA416, MA3491)**.
- Matrices act as **linear transformations** on vectors.
- **Linear transformations** on **vector spaces** are studied in **Linear Algebra (MA283)**.

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.

MA180-4

The Language of Mathematics: Logic and Sets.

Propositional Logic.
Valid Arguments.
Sets and Boolean Algebra.
Functions and Relations.
Summary.

Examples of Algebraic Objects: Permutations and Polynomials.

Composition of Functions.
Permutations.
Polynomials.
Factorisation of Polynomials.
Summary.

Mathematical Tools: Induction and Matrix Algebra.

Mathematical Induction.
Examples and Applications of Induction.
Determinants.
Eigenvalues and Eigenvectors.
Summary.

Course Summary and Outlook.